MATHEMATICS

B.Sc Part - I, Paper II

Topic - Tangent and normal (Pedal Equation)

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Pedal Equation

Pedal equation of a curve is a relation between p and r where p is the perpendicular from the pole on the tangent and r the radius vector r, the radius vector is the distance of any point on the curve from the origin (or pole).

1) To determine the pedal equation of a curve whose cartesian equation is given.

Let the cartesian equation of the curve be given by $y = f(x)$ ---- (1)

We know that the tangent at any point (x, y) to the curve $y = f(x)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

This tangent may also be written as

$$Y - y = X\frac{dy}{dx} - x\frac{dy}{dx}$$

$$\text{or, } X\frac{dy}{dx} - Y + \left(y - x\frac{dy}{dx}\right) = 0$$

$$\therefore p = \frac{y - x\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \quad \text{-----(2)}$$

($\because p$ be the length of the perpendicular from $(0,0)$ to this tangent)

Also we have

$$r^2 = x^2 + y^2 \quad \text{------.(3)} \quad \left[\because \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right]$$

Eliminating x, y between (1), (2) and (3) we obtain the required pedal equation of the curve $y = f(x)$.

If the equation of the curve be $f(x,y)=0$, then to find the pedal equation :

Since the equation of the curve be

$$f(x,y)=0 \qquad \text{-------- (1)}$$

Let us take the origin at the point with regard to the pedal equation is to be obtained.

The equation of the tangent at the point (x,y) on the curve is

$$xf_x + yf_y - (xf_x + yf_y) = 0.$$

If p be the perpendicular from the origin on it then

$$p^2 = \frac{(xf_x+yf_y)^2}{f_x^2+f_y^2} \qquad \text{------ (2)}$$

Also we have $\quad r^2 = x^2+y^2 \qquad$ ------ (3)

eliminating x and y from (1), (2) and (3) we get the required pedal equation.

2. To determine the Pedal equation of the curve whose polar equation is given.

Let us take the pole at the point with regard and to which the pedal equation is to be obtained and let the polar equation of the curve be given by $r=f(\theta)$ —(1)

let p be the perpendicular from the origin on the tangent at (r,θ) then we have

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \qquad \text{-------- (2)}$$

The pedal equation is obtained by eliminating θ between (1) and (2).

when the polar equation of the curve be $f(r,\theta)=0$, then to find the Pedal equation.

We have $f(r,\theta)=0$ ------ (1)

$$\tan\phi = r\frac{d\theta}{dr} \qquad \text{---- (2)}$$

and $\quad p = r\sin\phi \qquad$ ---- (3)

The pedal equation is obtained by eliminating θ and ϕ between (1), (2) and (3).

Example 1.

In the curve $r^m = a^m \cos m\theta$, Prove that

$$\frac{ds}{d\theta} = a\,(\sec m\theta)^{\frac{m-1}{m}}$$

We have $r^m = a^m \cos m\theta$

Taking log from both sides, we get

$$m \log r = m \log a + \log \cos m\theta.$$

Differentiating both sides w.r.t, θ, we get

$$m\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{1}{\cos m\theta}(-\sin m\theta).m = -\frac{m \sin m\theta}{\cos m\theta}$$

or $\dfrac{dr}{d\theta} = -r \tan m\theta$

Now $$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2 + r^2 \tan^2 m\theta}$$

$$= \sqrt{r^2 \sec^2 m\theta} = r \sec m\theta$$

$$\frac{ds}{d\theta} = r \sec m\theta = a\,(\cos m\theta)^{\frac{1}{m}}.\sec m\theta$$

$$[\because r^m = a^m \cos m\theta]$$

$$= a\,.\frac{\sec m\theta}{(\sec m\theta)^{\frac{1}{m}}} = a\,(\sec m\theta)^{1-\frac{1}{m}}$$

$$= a\,(\sec m\theta)^{\frac{m-1}{m}}$$

Example II. Show that in the curve $r = a^{\theta}$, the polar sub-tangent and the polar sub-normal have a constant ratio.

Differentiating this equation w.r.t. θ, we get

$$\frac{dr}{d\theta} = a^{\theta}, \log a$$

and therefore, $\dfrac{d\theta}{dr} = \dfrac{1}{a^{\theta} \log a}$

$$\frac{\text{The polar sub-tangent}}{\text{The polar sub-normal}} = \frac{r^2 \frac{d\theta}{dr}}{\frac{dr}{d\theta}}$$

$$= \frac{a^{2\theta} \cdot \frac{1}{a^\theta \log a}}{a^\theta \log a} = \frac{a^{2\theta}}{a^{2\theta}} \cdot \frac{1}{(\log a)^2}$$

$$\frac{\text{The polar sub-tangent}}{\text{The polar sub-normal}} = \frac{1}{(\log a)^2} = \text{constant.}$$

Example III. Show that the curves

$$r = a(1+\cos\theta),\ r = b(1-\cos\theta)$$

cut orthogonally.

We have the equation of the curves as

$$r = a(1+\cos\theta) \quad \text{---} (1)$$

$$r = b(1-\cos\theta) \quad \text{----} (2)$$

from (1) we have

$$r_1 = \frac{dr}{d\theta} = -a\sin\theta$$

$$\therefore \tan\phi_1 = \frac{r}{r_1} = \frac{r}{-a\sin\theta} = \frac{a(1+\cos\theta)}{-a\sin\theta} = \frac{-(1+\cos\theta)}{\sin\theta}$$

Again from (2), we have

$$\frac{dr}{d\theta} = b\sin\theta$$

$$\tan\phi = r\frac{d\theta}{dr} = \frac{b(1-\cos\theta)}{b\sin\theta}$$

$$\therefore \tan\phi_1 \tan\phi_2 = -\frac{(1+\cos\theta)}{\sin\theta} \cdot \frac{(1-\cos\theta)}{\sin\theta} = -\frac{(1-\cos^2\theta)}{\sin^2\theta}$$

$$= -1.$$

Hence the curves orthononally.

Example IV: Find the pedal equation of $r = a\theta$.

The curve is $r = a\theta$

Differentiating this w.r.t. θ we have

$$\frac{dr}{d\theta} = a.$$

But $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

or, $\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4} = \frac{r^2+a^2}{r^4}$

$$\therefore p^2 = \frac{r^4}{r^2+a^2}$$

This is a relation between p and r hence this is the required pedal equation.